Indian Statistical Institute
Bangalore Centre
Mid-Semester Examination
B. Math. Second Year
Statistical Methods II
Answer as much as you can. The maximum you can score is 60
Time :- 3 hours

1. (a) When is a statistic said to be **sufficient** for a parameter θ? State factorization theorem and prove it for discrete distributions.

   (b) Define **minimal sufficiency**. State and prove a result which helps to find a minimal sufficient statistics for a parameter θ.

   (c) Suppose $X_1, \cdots X_n$ are i.i.d random variables following the same distribution as X, which is given below. In each case find a **minimal sufficient statistic** for θ.

   (i) X follows Uniform $(\theta - 1/2, \theta + 1/2)$.

   (ii) The pd.f of X is $f(x|\theta) = exp[-(x-\theta)], \ -\infty < \theta < x < \infty$.

   [(2+5) + (2 + 4) + (3 x 2) = 19]

2. (a)When is a statistic said to be **complete** ?

   (b) Consider the following statistic $T(X)$ with the given family of distributions. In each case, prove or disprove the statement "the family of distributions of $T(X)$ is complete".

   (i) $T(X) = X$, X follows Poisson (λ), $\lambda = 2$ or 3.

   (ii) $T(X) = X_{(n)}, \ X_1, \cdots X_n$ are i.i.d random variables with p.d.f as follows.

   $$f(x|\theta) = 2x/(\theta^2), \ 0 < x < \theta, \theta > 0.$$

   [3 + 4 x 2 = 11]

3. (a)Define **maximum likelihood estimator (MLE)**.

   (b) In a town of Karnataka with population more than 100,000, there are five languages spoken. It was decided to estimate the proportions of residents speaking in different languages. A random sample of N residents was drawn and data on the language spoken was obtained. Show that the 'obvious' estimator is the MLE of the vector of population proportions.

   [2 + 7 = 9]

4. (a) Define **uniformly minimum variance unbiased estimator (UMVUE)** of a parameter.

   (b) Suppose X is a random variable with a p.d.f $f(x|\theta)$. Suppose the family of distributions $\{f(x|\theta), \ \theta \in \Theta\}$ satisfies the following conditions.

   (i) $\Theta \subset R$ and the support $\mathcal{X}$ of x is independent of θ.

(ii) $\frac{\partial}{\partial\theta}log[f(x|\theta)]$ exists and is finite $\forall\theta \in \Theta$ and $\forall x \in \mathcal{X}$.

(iii) For any statistic $T(X)$ with finite expectation $\forall\theta \in \Theta$, $\frac{d}{d\theta}E_\theta[T(X)] = \int_{\mathcal{X}} T(x)\frac{\partial}{\partial\theta}f(x|\theta)dx$.

Derive the Crammer-Rao lower bound for the variance of an unbiased estimator of θ.

(c) Suppose $X_1, \cdots X_n$ are i.i.d random variables with p.d.f as in Q2(b)(ii). Let $W(X) = cX_{(n)}$.

(i) Find c such that $W(X)$ is unbiased for θ.

(ii) Find the variance of $W(X)$.

(iii) See whether the variance of $W(X)$ satisfy Crammer-Rao lower bound. Justify what you see.

(d) State Rao-Blackwell Theorem. [Proof is not needed.]

(e) Suppose $T(X)$ is complete sufficient for θ and $W = \phi(T)$. Show that W is UMVUE for its expected value.

(f) Find the UMVUE for θ, where θ is as in Q(c).

[2 + 8 + 3 x 3 + 2 + 4 + 3 = 28]